THE

MATHEMATICAL GAZETTE.

EDITED BY

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THE DISSECTION OF RECTILINEAL FIGURES.

I AM not acquainted with any investigation of the problem of dissecting two given rectilineal figures of equal area, by straight lines, so that the parts of either will fit on the other, and the results given here are new to me. The problem is one which suggests a variety of interesting geometrical constructions. It is natural to consider first how, and in how many different ways, a dissection can be made so as to give the smallest possible number of parts; but types of dissection which do not give the smallest possible number of parts may also be worth attention. I exclude cases of dissections depending on some accidental or unspecified coincidence of dimensions in the given figures.

Let us first consider the case of a pair of triangles. If the triangles have common side, and do not differ too much in shape, there are two different dissections which are obtained by drawing in each triangle lines equal to

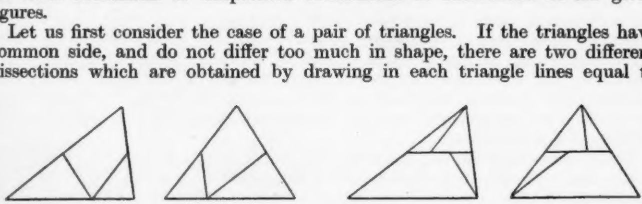


FIG. 1.

half sides of the other. These are shown in Fig. 1. The meaning of the condition that the triangles do not differ too much in shape may be seen by noticing that, if they are suitably placed with regard to one another, they can be projected orthogonally without the dissection being affected. One scheme gives a three-part, and the other, in which the angles of the triangles are divided, a four-part dissection. If the triangles differ too much in shape for these dissections to be possible, we can still get a dissection of each type, but with more parts, and with what I call broken lines in each triangle; that is to say, with parallel lines in each triangle making up half sides of the other. For the first type of dissection the triangles are to be placed with equal bases parallel to each other, and with two sides of each bisecting two sides of the

other. The dissection obtained is shown in Fig. 2. The second type of dissection is shown in Fig. 3, and is obtained by making the triangles into

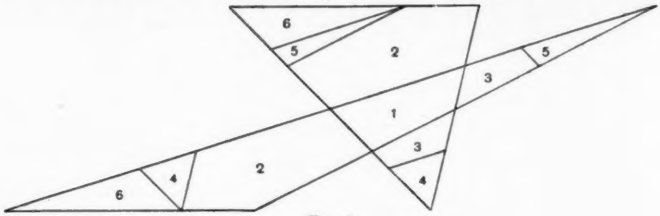


FIG. 2.

parallelograms on the same base and between the same parallels. In each case the procedure is fairly obvious. The three-part and four-part dissections are what I call the fundamental cases of these two types.

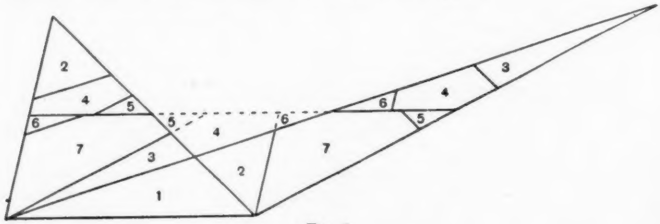


FIG. 3.

Accordingly we can dissect any given pair of triangles, of equal area, by constructing a third triangle, of the same area, with one side equal to a side of one triangle and another side equal to a side of the other triangle, and by dissecting this third triangle so that the parts fit on to each of the given triangles. In this way two types of dissection are obtained from the constructions shown in Fig. 1. One of them does not divide the angles of the triangles, and gives, as the fundamental case, a four-part dissection for triangles not differing too much in shape; the other divides the angles, and gives, as the fundamental case, a seven-part dissection for triangles not differing too much in shape.

These dissections having been established, it is not necessary, in order to study them, to make any further use of the method by which they have been obtained. The four-part dissection of a pair of triangles, ABC and DEF, is one in which each triangle is divided by three lines equal respectively to

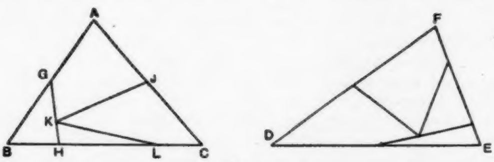


FIG. 4.

the half sides of the other triangle. The dissection of ABC is made as follows, as shown in Fig. 4. From the middle point of AB draw GH equal to $\frac{1}{2}EF$;

from the middle point of AC draw JK equal to $\frac{1}{2}FD$; and take HL equal to $\frac{1}{2}BC$. Then JL is equal and parallel to GH, and the area of a triangle JKL is a quarter of ABC; therefore the angle LJK is either equal or supplementary to F. Suppose the figure to be drawn so that this angle is equal to F, then KLJ is similar to DEF, and KL is equal to $\frac{1}{2}DE$. Thus the angles D, E, F are formed at K. The dissection of DEF is made in a similar way, starting at the middle point of DE. The character of the dissection and the relations between the parts are shown in Fig. 5, in which dotted lines are added to show

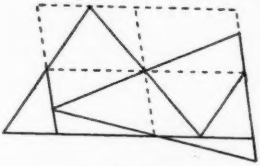


Fig. 5.

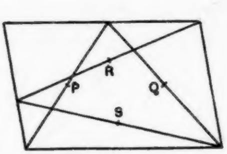


Fig. 6.

the construction of parallelograms of area equal to half that of each triangle, and with one pair of sides equal to a half side of one triangle, and the other pair of sides equal to a half side of the other triangle. It is clear from this figure that the condition that the four-part dissection exists is that a parallelogram can be drawn within which both the triangles can be placed, each having a side of the parallelogram for its base, and its vertex in the opposite side. Draw such a parallelogram and the triangles within it, as in Fig. 6, and let P, Q, R, S be the middle points of their sides. Examination of this figure shows that we obtain from any such parallelogram two distinct four-part dissections, and no more. One is obtained by shifting one of the triangles, without rotation, so as to bring R and Q into coincidence, and the other by a similar shift bringing P and S into coincidence. Thus a pair of triangles which has one, and only one, parallelogram satisfying our condition, has two four-part dissections. There cannot be a pair of triangles with only one four-part dissection. The pair of triangles drawn in Figs. 4, 5 and 6 has four four-part dissections.

The greatest number of four-part dissections which a pair of triangles can have is twelve. This occurs when both triangles are acute angled, and each side of either is greater than any of the perpendiculars from angles to opposite sides of the other. This may be proved by considering the different cases that can arise. It is obvious that triangles with the greatest number of dissections must approach to being equilateral. Let us therefore consider all possible pairs of triangles of equal area and nearly equilateral. There are 20 different possible arrangements of order of magnitude of the sides; and by separate examination of all these 20 pairs of triangles, it is found that each of them has exactly twelve distinct four-part dissections. We have now to consider at what point this can cease to be true as the triangles recede from being nearly equilateral. The question of the order of magnitude of the sides has been provided for, so the first crisis in the affairs of our 20 pairs of triangles, which can affect our reckoning, happens when a half side of one of the triangles of a pair becomes less than a half perpendicular of the other. It seems clear also that when this takes place a four-part dissection must be lost without compensation.

A considerable range of pairs of equal triangles with no four-part dissection, as well as those which have four-part dissections, have at least one five-part dissection of the same type, with one broken line. Such a pair is shown in Fig. 7. Here the triangles ABC, DEF, and a five-part dissection of them, are drawn in full lines, with corresponding lines parallel; and all other lines of the figure are dotted. The dissection shown here is given by

drawing GJ and HK, from the middle points of BC and EF, equal respectively to $\frac{1}{2}FD$ and $\frac{1}{2}AB$. Let L and M be the middle points of CA and DE, then the areas of GJL and HMK are equal, each being a quarter of ABC or DEF; therefore the angles LGJ and KHM are equal or supplementary. The case which concerns us is that in which the figure is drawn so that these triangles are congruent. Make the cuts GJ and HK, and place the parts $GJAC$ and HKF in the positions $GNOB$ and HPE. Then we have equal quadrilaterals

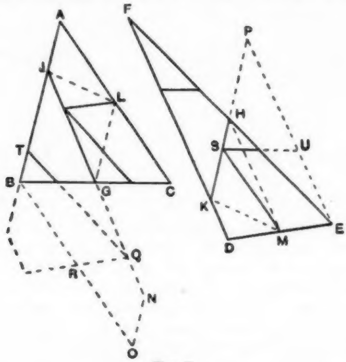


FIG. 7.

$JBON$ and $PKDE$, in which the angles at J and P are equal, one of them being equal to LGJ and the other to KHM. By one more cut, either QR, equal and parallel to EM, or MS, equal and parallel to CL, either figure can be made identical with the other. But to make the parts the same we have to make a third cut in each figure, giving a broken line in each triangle, namely, QT to provide for EH and SU to provide for BG. The general character of the way in which broken lines are to be placed in each triangle, and the angles of each triangle are formed in the other, is indicated in this example, and it is not generally necessary to follow a series of steps, like those by means of which a proof of this dissection has been indicated. There is a great variety of different cases; for example, Fig. 8 shows a six-part dissection of the same

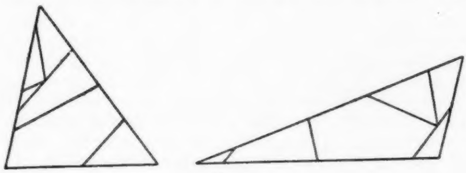


FIG. 8.

type, and of the same pair of triangles. For this a step by step proof might be given, quadrilaterals being formed as before. Here the first cut in each triangle is along a broken line. It is necessary to pay attention to the rejection of constructions such as would give a dissection of one triangle suitable for the other triangle reversed, or giving an angle supplementary to one of those which are required.

For triangles differing much in shape any number of parts may be needed, and the lines of section may be broken into any number of pieces. It is always

possible to place in one or other of the triangles a line drawn from the middle point of a side, equal to half a side of the other one; but a little care is needed to see whether this leads to a true dissection, and we cannot in this way get all varieties of dissections of the type under consideration. The number of parts needed for dealing with a pair of triangles, such that the greatest side of one is about three times the greatest side of the other, is about eight.

If the plan is adopted of dividing the angles of each triangle to make up angles of the other, the angles must be divided into at least five pieces; and it seems improbable that there is any economical dissection of this kind dividing the angles into less than six pieces. The seven-part dissection which has been mentioned above is shown in Fig. 9, in which the triangles are placed

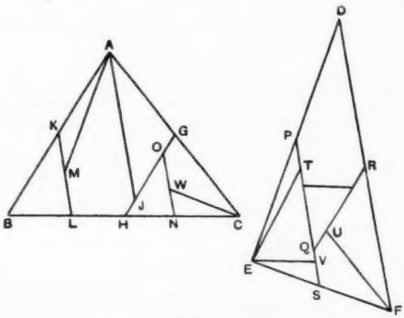


FIG. 9.

with corresponding lines parallel. Here we have to distinguish between the two triangles, as their dissections are different; the triangle ABC must be properly chosen to have the dissection drawn in the figure, and the angle A to be divided into three must be properly chosen. To make the construction, draw GH joining the middle points of AC and BC, and AJ equal to $\frac{1}{2}DF$, and KL parallel to AJ through the middle point of AB. Then, if M is any point on KL, the area of a triangle AMJ is a quarter of ABC. Take AM equal to $\frac{1}{2}DE$, then the angle MAJ must be equal or supplementary to D. For success it must be equal to D; if it proved to be supplementary, the attempted construction would be abandoned, and a fresh start would be made in some other way. Take HN equal to LH, and draw NO parallel to AJ; then $AMLHJ$ and HNO make up a quadrilateral $DPQR$, PQ being parallel to DF, so that produced it bisects EF in S. Draw ET equal to $\frac{1}{2}AB$; then the angles at P and M being equal, the triangles AKM and ETP are congruent if the construction is succeeding, and ET is parallel to QR. Draw FU equal to $\frac{1}{2}AC$; then in the same way FUR and AGJ are congruent. Now two cases arise according as the angle TES is greater or less than B. In the figure it is greater than B, so by drawing EV equal to BL, we have, by due choice among alternatives, TEV and KBL congruent, the angle T having been made equal to K. Finally CW is drawn equal to $\frac{1}{2}EF$. In the other case a line would be drawn from B equal to $\frac{1}{2}EF$. The completion of the proof of this dissection, subject to steering a course among alternatives in the construction, presents no difficulty. The seven-part dissection is the fundamental case of this type. If the triangles differ too much in shape for this to be possible, dissections of the same type, but with more parts, and broken lines, can be made, as in the case of the four-part type of dissection.

For the dissection of a rectangle and a square, the construction shown in

Fig. 10 is universally applicable. The fundamental case is a three-part dissection. This occurs when the longest side of the rectangle is less than the diagonal of the square. The figure shows also a case in which we get four parts and a broken line. The same dissection applies to any pair of parallelograms of equal area as is shown in Fig. 11, *GBCF* and *JBEH* being the parallelograms in question. Each parallelogram is divided by two lines equal to the sides of the other. Now these parallelograms can be dissected so as to form a pair of triangles *ABC* and *DBE*; and thus we get, as shown in Fig. 11,

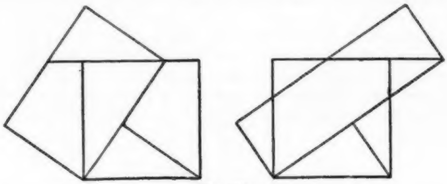


FIG. 10.

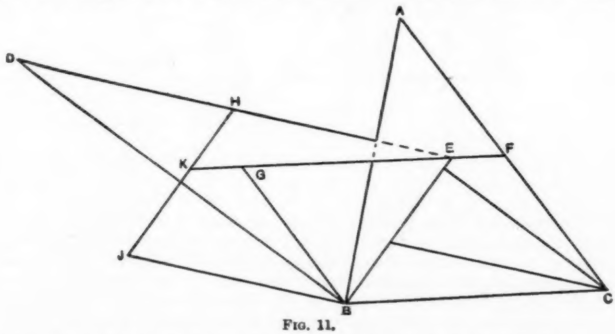


FIG. 11.

a dissection of a pair of triangles, in which the angles of each are divided to make up the angles of the other. The fundamental case of this is a five-part dissection as drawn in the figure. This dissection depends on a line, such as *BE* or *EK*, being drawn in each triangle equal to a whole side of the other one. In many cases of a pair of triangles no dissection of any other type exists more economical of parts than this one. The pair of triangles shown in the figure has no four-part dissection, but has several five-part dissections of the four-part type.

There are several types of dissection of a triangle and a square of the same area. Fig. 12 shows a type of dissection which is applicable to all cases in which the triangle has a side greater than twice the side of the square. A line equal to half this side of the triangle is drawn from a corner of the square to a side, or side produced, of the square. The other two sides of the triangle are bisected by sides, or sides produced, of the square. The fundamental case is a four-part dissection. In other cases we get more parts and broken lines. The figure shows a five-part dissection. The condition for a four-part dissection is that the triangle can be drawn within a square of four times its area, with one angular point at the middle point of a side of this square, and another angular point on the opposite side.

Another type of dissection of a triangle and a square is derived from the

construction shown in Figs. 10 and 11. It is obtained by drawing a line equal to a side of the triangle from a corner of the square to a side, or a side produced, of the square. The fundamental case is the four-part dissection

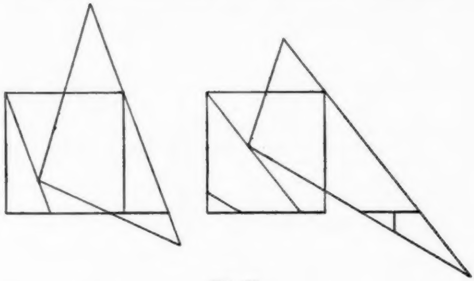


FIG. 12.

which is shown in Fig. 13. The condition for the existence of this dissection is that the triangle can be placed within a square of four times its area, with one angle at a corner of the square, and the opposite side produced passing through one of the adjacent corners; and it has four parts if this side is less

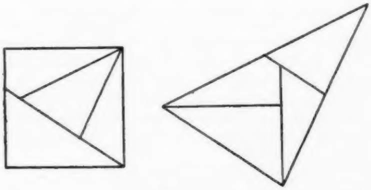


FIG. 13.

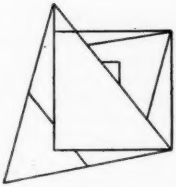


FIG. 14.

than half the diagonal of the square. When the latter condition is not fulfilled we get a dissection with more parts. When the triangle approaches to being equilateral, we get the five-part dissection shown in Fig. 14. I have not been able to find any type of dissection which gives four parts for a triangle which approaches to being equilateral.

Every triangle has either one or the other of these two types of dissection of a triangle and a square, and some triangles have both. For example, the triangle whose sides are 86, 70 and 45 has two dissections of the first type, and one dissection of the second type, and each of these has four parts.

The question of the dissection of a pair of quadrilaterals of equal area affords a good deal of scope for ingenuity. The case of re-entrant quadrilaterals does not appear to need any special treatment. By a four-part dissection, by lines bisecting the sides, any quadrilateral can be made into a parallelogram with one side equal and parallel to one diagonal, and the other side equal and parallel to half the other diagonal. This treatment of each quadrilateral gives a type of dissection for the pair, the fundamental case of which, shown in Fig. 15, has eleven parts. In this figure the quadrilaterals and their dissection are drawn with full lines, and other lines of the construction are dotted. The quadrilaterals are placed so that corresponding lines are parallel. The procedure adopted is to obtain a dissection of the pair of parallelograms *ABCD* and *EFGH*. To do this *EL* and *CJ* are drawn equal respectively to

AD and EF, and points M and K are taken so that FM and BK are equal respectively to AB and FG. The rest of the construction is obvious. The angles of the two quadrilaterals are formed at N and P respectively. I have not found out how to define the range of cases in which this type of dissection

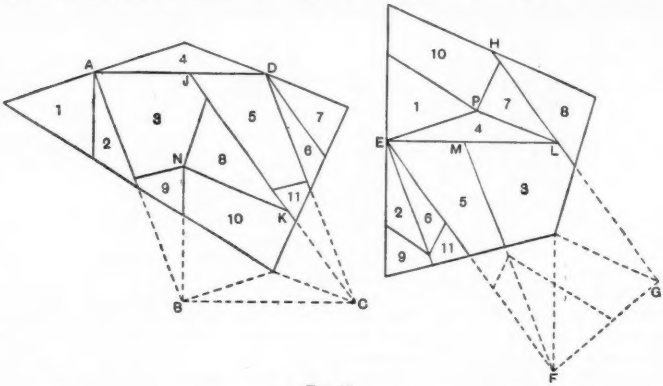


FIG. 15.

gives no more than eleven parts. That this is not the only type of dissection of a pair of quadrilaterals, in which what appears to be the fundamental case gives eleven parts, may be seen as follows. There are at least two types (which can be found without much difficulty) of seven-part dissection of a quadrilateral and a triangle. Take a pair of quadrilaterals of equal area, and dimensions within a suitable range, and divide one of them into two triangles, and divide the other into a triangle equal to one of these triangles

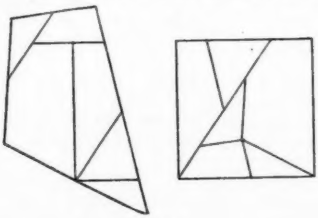


FIG. 16.

and a quadrilateral equal to the other triangle. Now make a four-part dissection of the pair of equal triangles, and a seven-part dissection of a triangle and the quadrilateral which is equal to it; in this way we get an eleven-part dissection of the original pair of quadrilaterals, in which some of their angles are divided.

The dissection of a quadrilateral and a square can be made in several different ways, and appears to require at least six parts. One six-part dissection can be obtained by drawing from one angle of the square a line to one of the opposite sides equal to a diagonal of the quadrilateral, and completing the construction in the way shown in Fig. 16. The angles of the quadrilateral are formed at a point within the square.

W. H. MACAULAY.

LINEAR DYNAMICS.

THE *Note on Momentum and Kinetic Energy*—432. [6. a. β.]—by Mr. W. M. Roberts, is interesting in showing a return in the Royal Military Academy to the gravitation unit of force in Dynamics, as more tangible for application to practical gunnery and engineering than the absolute system, introduced into theoretical instruction about 40 years, and described in Maxwell's *Matter and Motion.*

I wish to recommend a grouping of the algebraical symbols in his equations, in such a manner as to bring out the physical meaning, by writing them

$$Ps = W\frac{v^2-u^2}{2g}, \quad Pt = W\frac{v-u}{g};$$

and then P and W are measured in the same unit, lb or ton; while s and $\frac{v^2-u^2}{2g}$ represent a length in feet, t and $\frac{v-u}{g}$ a time in seconds, v and u being a velocity in feet per second (f/s).

In a field of gravity g(f/s^2), $\frac{v^2-u^2}{2g}$ is the vertical descent of a body falling freely, in the time $\frac{v-u}{g}$ seconds, during which the velocity grows from u to v, f/s; and in a practical problem it is generally close enough to take $g=32$.

It is important that g should not stray from its proper place, or take cover under W; or else the old mystification comes in again, where $\frac{W}{g}$ has been replaced by a single letter, M; a mere lazy algebraical device at first, to save trouble in writing and printing, but exalted 40 years ago into a profound dynamical principle, to the confusion of every novice in Dynamics.

Invited to give an address to the Mathematical Club at Cambridge, I chose as the title—*A lecture of one minute on Linear Dynamics*; intended to show how the theory and abstract formulas would go on to a sheet of note paper, enough to carry the beginner through his first year in Dynamics, with application to familiar sights and experiments.

A force F, pounds or tons, acting for t seconds through s feet on a body of weight W, lb or ton (a railway train, tramcar, motibus, bicycle, boat, steamer, shot in a gun, ...) will give it velocity v f/s from rest, such that

$$\text{(A)} \quad \underset{\text{(lb or ton) (sec.)}}{F \times t} = \underset{\text{(lb or ton) (sec.)}}{W \times \frac{v}{g}},$$ momentum, in lb- or ton-seconds, or second-lb or ton;

$$\text{(B)} \quad \underset{\text{(lb or ton) (feet)}}{F \times s} = \underset{\text{(lb or ton) (feet)}}{W \times \frac{v^2}{2g}},$$ energy, in lb- or ton-feet, or ft-lb or ton;

and falling freely in a field of gravity g, where $F=W$, v is acquired in falling for $\frac{v}{g}$ seconds through $\frac{v^2}{2g}$ feet vertical.

We say, generally, ft-lb or ft-ton, for euphony, in preference to lb-ft or ton-ft; the combination word of lb- or ton-second, or second-lb or ton, has not yet come into general use, and so the choice is optional.

(*Math. Gazette*, Jan. 1913, p. 11.)

The dynamical quantity $W\frac{v^2}{2g}$, the *energy*, is the equivalent of Fs, the *work*; and $W\frac{v}{g}$, the *momentum*, is the equivalent of Ft, which may be called

the *impulse.* As the time t during which F acts grows smaller, the impulse changes from a *push* to a *blow,* the equivalent momentum remaining the same if F increases in proportion, so as to keep Ft unaltered.

Between (A) and (B) we obtain by division

(C) $\frac{s}{t}=\frac{1}{2}v$, the *average* velocity.

The principle of momentum is given in (A) first, as more intelligible to the innate notions of a beginner; and it is a definite translation into an equation of Newton's Second Law of Motion. Change of motion (quantity of motion or momentum) is proportional to the impressed force (the change being measured per second).

Then (B) and (C) may follow, in either order, whichever appeals to the logical sense of the learner.

It will be noticed that the idea of acceleration has been kept out of sight, except in the inevitable g of the gravitation system. It is the usual experience that a grasp of the meaning of acceleration does not arrive till after some progress has been made with the applications of the principle; and it is notorious that the engine driver, whose business is the making of acceleration, has a very vague idea of its measure.

But with acceleration a defined as the growth of velocity per second, we can rewrite (A) as

(D) $$a=\frac{v}{t}=\frac{P}{W}g,$$

and so arrive at the equation with which Mr. Roberts begins.

To complete the history of a dynamical problem, the body must be brought to rest again, as in a run AD from one station to the next; and if this is done in CD by a break resistance B, acting for t' seconds through s' feet,

(A′) $$Ft=W\frac{v}{g}=Bt',$$

(B′) $$Fs=W\frac{v^2}{2g}=Bs',$$

(C′) $$\frac{s}{t}=\frac{1}{2}v=\frac{s'}{t'}.$$

The middle part of the journey BC may be supposed run at full speed v, and then a graphical representation is given, in Fig. 1, of a simple journey

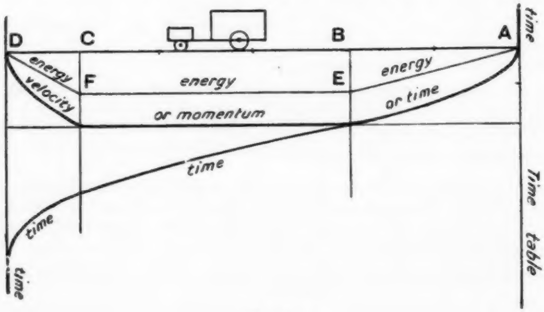


FIG. 1.

from rest to rest; as a graphical time table, except that in Bradshaw, the distance line AD of the stations is in the vertical margin to the left, and time stretches across the page.

On a horizontal road the carriage moves as if the vertical field of gravity g was disturbed by a horizontal field, $\frac{F}{W}g$ or $\frac{B}{W}g$; and the direction of resultant apparent gravity will be given by a short plumb line, which sets itself at right angles to the line $AEFD$ of apparent level, or by the angle at which a man must stand up on his feet, without other support; making in Fig. 1,

$$\frac{EB}{BA}=\frac{F}{W}, \quad \frac{FC}{CD}=\frac{B}{W}.$$

In the tube railway the sensation of change in apparent gravity is noticeable to the straphanger, and to the passenger walking along the car, according as he is entering or leaving by the door at the front or rear end; and the jerk he feels is due to the sudden change in the direction of apparent gravity when the car stops or starts.

The sudden change is felt very strong on the top of the motibus, according as the driver inserts the clutch, or applies the break.

We have postulated the kinematical theorems of a body falling freely under gravity g, in saying that $\frac{v}{g}$ is the number of seconds falling to acquire velocity v from rest; and this follows from (D), when $a=g$,

$$(1) \quad g=\frac{v}{t}, \quad t=\frac{v}{g},$$

and g is constant, as enunciated by Galileo.

Then, working with (c), the average velocity principle,

$$(2) \quad s=\tfrac{1}{2}vt=\frac{v^2}{2g},$$

the depth s vertical to fall to acquire velocity v from rest.

Or, if the body was already falling with velocity u,

$$(3) \quad g=\frac{v-u}{t}, \quad t=\frac{v-u}{g},$$

$$(4) \quad s=\tfrac{1}{2}(v+u)t=\frac{v^2-u^2}{2g},$$

assuming the average velocity $\frac{1}{2}(v+u)$.

So also the gain of energy, $W\frac{v^2-u^2}{2g}$, composed of the factors $W\frac{v-u}{g}$ and $\frac{1}{2}(v+u)$, is seen to be the work done by the impulse $W\frac{v-u}{g}$ multiplied by the mean average velocity $\frac{1}{2}(v+u)$.

In a different notation, if the velocity grows from $V-\frac{1}{2}U$ to $V+\frac{1}{2}U$ through a mean velocity U, we can rewrite Mr. Roberts's equations

$$Ps=W\frac{Uv}{g}, \quad Pt=W\frac{U}{g}.$$

Here is abstract theory enough to serve for the beginner in Dynamics, for six months or a year according to ability; and it will give a definite grasp of the laws of motion, if practised on concrete tangible applications, which in Kelvin's opinion should be chiefly numerical, and on a large scale, such as I have collected in my *Notes on Dynamics*.

Next, by assuming the law of the spring, the alteration of trim can be calculated of the floor of the carriage as the break is applied or the clutch inserted; and without leaving Linear Dynamics the motion can be investigated of the body of the carriage, making a simple vertical vibration on the springs, the force of restitution and the acceleration being proportional to the displacement from the position of equilibrium O.

Variable force and acceleration is then introduced, so that in the most general case we can rewrite (A) and (B)

$$(5)\quad W\frac{v}{g}=\int F\,dt,\quad W\frac{v^2}{2g}=\int F\,ds,$$

according as F is given as a function of time t or distance s.

With an acceleration n^2s to O, the force required to move the weight W is $F=-W\frac{n^2s}{g}$; and assuming the law of the average, as in (c), the average value of s between a and s is $\frac{1}{2}(a+s)$, and acting through $a-s$ feet, the average force $W\frac{n^2}{g}\frac{1}{2}(a+s)$ will do $W\frac{n^2}{g}\frac{1}{2}(a^2-s^2)$ ft-lb or -ton of work, to be equated to the energy $W\frac{v^2}{2g}$, if the body starts from rest at A, a distance a from O.

Then, in Fig. 2,

$$(6)\quad v^2=n^2(a^2-s^2)=n^2.\,AM.\,MA',$$

and v is n times the G.M. of the distances from A and A' in an oscillation AA, to an equal distance on each side of O.

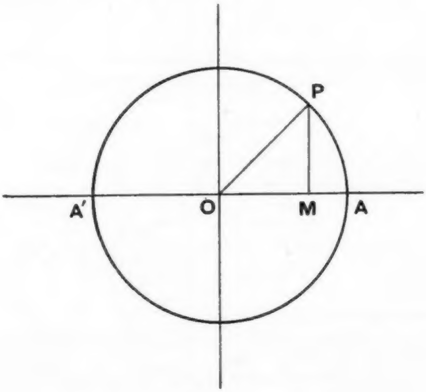


Fig. 2.

In Maxwell's manner (*Matter and Motion*) draw the circle APA' on the diameter AA' with the ordinate MP; then, the velocity of M being

$$n\sqrt{(AM.\,MA')}=n.\,MP,$$

the velocity of P in the triangle of velocity OMP is $n.\,OP$, a constant; and the arc AP described in time t is nta, making

$$(7)\quad s=a\cos nt,\quad v=na\sin nt;$$

and M in AA' follows P on the circle, as if looked at in a direction perpendicular to AA'.

Treating the horizontal oscillation of the simple pendulum in the same way as the spring in vertical vibration, replacing n^2 by $\frac{g}{l}$, and supposing a small compared with l, it is found that the equivalent pendulum length l is the vertical set of the springs, when the body sinks down to rest on them.

The angular oscillation of the body, in pitching and rolling on the springs, must be reserved till after the consideration of moment of inertia, and simple motion of rotation, as of a flywheel about its axle, a door on hinges, or a carriage on wheels.

But time will be gained by following the method of Worthington's *Dynamics of Rotation*, in taking the theory for granted where W, v, F, and s, in (A) and (B) for linear dynamics, are replaced in angular motion by Wk^2 moment of inertia in lb-ft^2, ω angular velocity in radians per second, couple N in ft-lb, and angle θ in radians; applying the formulas to numerical examples, while the proof is reserved till later.

The next stage would be the consideration of the composition of force and velocity, and the distinction of Mr. Roberts' note between momentum as a vector and scalar energy.

But attention may be directed beforehand to the experimental facts of motion on a moving platform, in uniform velocity; the deck of a steamer or interior of a railway carriage, where the results of Linear Dynamics are found appreciably to be unaffected.

We are not considering the non-Newtonian Dynamics of Relativity, any more than non-Euclidean Geometry. Although it is useful for the teacher to have some acquaintance with these mysteries, better keep it dark from a beginner.

G. GREENHILL.

NOTE ON DESARGUES' THEOREM.

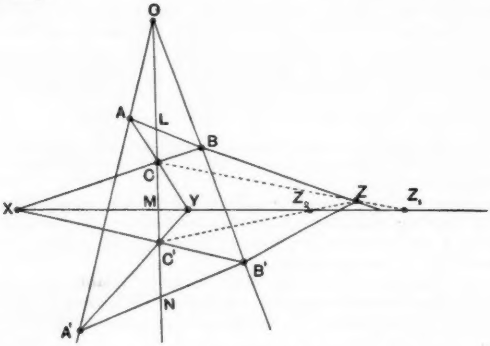
MAJOR DIXON has given in the *Gazette* for May an interesting representation, suggested, perhaps, by Hilbert's example of a non-desarguesian geometry, of a plane geometry, in which straight lines are represented by closed curves on a closed convex surface devoid of singularities. Employing his theory of order and "collation," he constructs this geometry so as to be desarguesian.

There are some points, however, in his discussion which are apt, in my opinion, to lead to misunderstanding, and in which he disguises rather than makes clear the fundamental assumptions involved. In the first place, the statement that Hilbert's definition of a straight line is inadequate is misleading. What appears to be intended is that Hilbert's first two groups of axioms are "non-categorical," *i.e.* they admit of essentially different (not isomorphic) representations in which the axioms of these groups are all satisfied; in one representation, for example, Desargues' theorem may be always true, in another it may be sometimes false. As a matter of fact, the straight line is never completely defined, since there always exist different, but isomorphic, representations of the elements of any system of geometry. Euclidean plane geometry, for example, can be equally well represented by the geometry of circles through a point in euclidean, elliptic or hyperbolic space, or by the geometry of horocycles on a horosphere in hyperbolic space.

The second point is that the impression is given that, according to Hilbert, Desargues' Theorem depends upon the axioms of congruence. If Hilbert has produced this impression it is unfortunate, and Major Dixon has done well to show that this theorem, which, as Hilbert states, is "ein ebener Schnittpunktsatz," can be established without any axioms of congruence, and without going outside the plane. My objection, however, is that in proving Desargues' Theorem, he has assumed much more than is contained in the theorem itself. He has, in fact, assumed the Fundamental Theorem of Projective Geometry,

viz. "if a one-to-one correspondence is established between the points of two lines in such a way that three points A, B, C of the one line are definitely correlated to three points A', B', C' of the other line, then in whatever way the correspondence is effected, the correspondent of any point P is always the same." He has assumed this explicitly in one of his earlier papers on "Order defined by Boundaries" (*Math. Gazette*, vol. vi. p. 173).

It is not surprising that Desargues' Theorem should be "proved" from this assumption. Put in more familiar terms, Major Dixon's proof is equivalent to the following.



ABC and $A'B'C'$ are triangles such that AA', BB', CC' are concurrent in O. BC, $B'C'$ cut in X; CA, $C'A'$ in Y; AB, $A'B'$ in Z. CZ and $C'Z$ cut XY in Z_1 and Z_2. OCC' cuts AB, XY, $A'B'$ in L, M, N.

Then, using the ordinary notation for perspectivity,

$$(XYMZ_1)\overline{\wedge}_C(BALZ)\overline{\wedge}_O(B'A'NZ)\overline{\wedge}_{C'}(XYMZ_2).$$

Therefore $Z_1=Z_2=Z$ by the Fundamental Theorem, and X, Y, Z are collinear.

Thus Desargues' Theorem is deduced from the Fundamental Theorem.* But the Fundamental Theorem is a capacious assumption to start from in establishing Desargues' Theorem. The Fundamental Theorem in its full extent involves the uniqueness of the Quadrilateral Construction and also Continuity. Desargues' Theorem does not involve Continuity, and is itself required in establishing the uniqueness of the Quadrilateral Construction. To assume the Fundamental Theorem in order to prove Desargues' Theorem is analogous to assuming the formulae of coordinate geometry in order to prove the propositions of the First Book of Euclid. D. M. Y. Sommerville.

A PLEA FOR ASTRONOMY.†

My first duty is to thank you for the honour you have conferred upon me in electing me as your President for the year. Once all mathematicians were astronomers and all astronomers were mathematicians. But now only some astronomers are mathematicians, and I fear that I can only claim to be one to a very limited extent. It was, however, my good fortune to be a member of a college where your ex-President was a lecturer, and to have learned there, from him and others of a

* Essentially the same form of proof is given by J. L. S. Hatton, *Projective Geometry* (Cam. Univ. Press, 1913), p. 19. The elegance of the proof disguises its logical unsoundness.

† Address to the London Branch of the Mathematical Association. February 7th, 1914.

very distinguished society, something of the pleasure to be obtained from the study of Mathematics.

I should like to-day to put in a plea on behalf of Astronomy to those of you who are engaged in the teaching of Mathematics. Although many people know something of the results of astronomical research, there is much ignorance of the every-day phenomena. Everybody has been taught that the earth turns about its axis, but many hardly realise that the stars rise and set. Similarly, it is believed by most people that the earth turns round the sun, but only a few know that different stars are visible at different times of the year. The succession of day and night, summer and winter are taken as a matter of course, but the earth's rotation about its axis and revolution round the sun are things that have been read in a book, or that we have heard men say. It certainly is not exaggeration to say that many well-educated people do not appreciate in the least the connection between the every-day phenomena with which they are familiar and the orthodox astronomical creed they have been taught to believe.

Some of you may say—This is true enough and a great pity, but that is not our fault. It would never do for us to give snippets of Astronomy instead of careful and well-considered courses of lessons in Algebra and Geometry and Trigonometry. It rests with the teachers of Geography rather than with those who teach Mathematics. I gladly recognise, from the limited experience I have had, that the teaching of Geography has improved immensely, and that the elementary ideas of the earth's size and shape and movements are taught in some schools in a very intelligible and proper manner. I am in absolute agreement with the view that it is the greatest mistake to teach a little of this and a little of that. But Astronomy and Mathematics have grown up together, and, I think, astronomical illustrations can sometimes be found which will enforce and perhaps enliven lessons in Geometry and Trigonometry. To take the first example that comes to one's mind—If a boy is told that the angular diameter of the sun is approximately half a degree, a definiteness is given to his conceptions of small angles, and his idea of a few minutes of angle is as definite as his idea of 30° or 45°.

The history of Astronomy, and the methods by which our knowledge of the size and shape of the earth were obtained and the distances of the planets determined, and the way in which the Copernican system was established, all furnish excellent illustrations of geometrical principles. Astronomy grew up with Geometry, and, using the word in the sense in which the word is used in France, a great part of Astronomy is Geometry. It was the clear conception of elementary geometrical principles, and the imagination and intellectual boldness which did not shrink from applying these principles to the stars, which led to the fundamental discoveries of Astronomy. Take, for example, the size and shape of the earth. The Greek astronomers inferred that the earth was spherical, from the fact that its shadow cast on the moon always had a circular edge. To a geometer the proof is as good a one as he needs. But it was really a great achievement to realise that the obscuration of the moon was merely the shadow of the earth cast by the sun, and had not some mysterious out-of-the-way explanation. I once had the pleasure of looking at a lunar eclipse with two boys. They were very much interested, and readily understood how the comparison of the size of the shadow with that of the moon gives the relative sizes of earth and moon, and thus the distance of the moon. Any one who sees an eclipse, and thinks about it, cannot fail to be impressed, and to feel that Geometry has some very interesting applications.

As another illustration, take the determination of the size of the earth which Eratosthenes made 200 years before Christ. You remember he found that at midsummer at midday the sun was vertically overhead at Assouan, but at Alexandria was inclined $7\frac{1}{2}°$ to the south. He measured the distance from Assouan to Alexandria, and thus calculated the earth's radius. Surely this is a very excellent example of the relations between angles and arcs, and has the merit of showing that they apply to big circles like a meridian of the earth as well as to small ones drawn on a sheet of paper.

To those of you who are unacquainted with it, I should like to recommend Mr. Berry's *History of Astronomy.* It was written as a University Extension Manual, and gives a very clear account of the way in which Astronomy grew up. The skill of the Greeks harmonised the facts with which they were acquainted into a science, and with epicycle and eccentric constructed formulae for the apparent movements of the heavenly bodies. The imagination of Copernicus did not shrink from simplifying the geometry of these movements by an hypothesis of the rotation of the earth and its revolution round the sun. Galileo with his telescope adduced arguments which appealed to everybody and not only to the mathematicians. Finally, the laws of motion and the foundation of Dynamics arose from the need to give a rational explanation of the movements of the planets round the sun.

When we have arrived as far as Dynamics, Astronomy naturally comes in for its share of attention, as it provides the simplest examples of orbital motion. But, if I may say so, there is sometimes among mathematicians a tendency to take the observations for granted and to treat Astronomy in a somewhat abstract manner. The planets move in ellipses, according to Kepler's laws, and the rest is a matter of pure reasoning. I should like to suggest that there is a great deal of interest in occasionally looking at the stars themselves, and seeing for oneself some of the facts which have led to the great generalisations of Astronomy. Perhaps I am wrong, and most of you have a look at the sky sometimes, and know what stars are to be seen at different times of the year, and notice the positions and movements of Venus or Saturn. But if any of you do not (and the frequency of cloudy nights may well be pleaded as an excuse), I would recommend learning the positions of a few of the brighter stars, watching the movement of the moon from night to night with reference to the sun and stars, and the movements of the brighter planets. You will, I am sure, in this way be filled with admiration for the discoveries of the old astronomers, and will appreciate the great step taken by Copernicus and Galileo in substituting the heliocentric for the geocentric system.

I should not have ventured to make these suggestions had it not been for the experience I had in teaching Astronomy at Edinburgh. It was my good fortune to have in my class some of the best mathematical students in the University. Naturally, my course was framed to fit in with and supplement what they had learned in the mathematical and physical classes. But what was of greatest interest to the pupils I had, was, I think, the historical part, and making them acquainted with the ordinary every-day phenomena, supplemented, it is true, by occasional peeps through the telescopes at Jupiter or Saturn's rings, or whatever was on view at the time. A number of these students have since become teachers, and I have been very pleased by the receipt of an occasional letter which showed that their interest in Astronomy was maintained, and that they found it useful to them in their profession.

As you have been good enough to elect me your President, I shall venture to give you a very brief summary of the trend of Astronomy in recent years. About sixty or seventy years ago, let us say after the discovery of Neptune, the view was largely held that the great discoveries in Astronomy had all been made. For a century and a half the greatest mathematicians had been engaged in deducing all the intricate movements of the planets from the simple law of gravitation. It was supposed that the outstanding questions were of minor importance, and that future astronomers would have to elaborate and work out in detail their predecessors' results.

Then came the new discoveries in Physics. The spectroscope was applied to determine the chemical and physical condition of the sun and stars. This has opened a new chapter in Astronomy. The determination of the gaseous constitution of some of the nebulae, the evidence that stars are in different stages of development, were among the earlier results. Comparatively recently great success has attended researches on the sun, and now the composition of different layers of the sun's atmosphere and movements in them are studied. The existence of a strong magnetic field in sun-spots has been experimentally proved, and of a weaker magnetic field over the whole disc of the sun.

But it has not been only in this new field that discoveries have been made during the last fifty years. The theory of the moon had been studied by Newton, Clairaut, d'Alembert, Euler, Laplace, Damoiseau, Plana, Poisson, Lubbock, Pontécoulant. Their theories were superseded in the middle of the last century by the two most exhaustive theories of Hansen and Delaunay—Hansen's being to some extent a numerical method, and with the practical aim of predicting accurate positions of the moon for the Nautical Almanacs, and Delaunay's one of remarkable theoretical elegance and completeness. Nevertheless, late in the last century, an American astronomer, Mr. G. W. Hill, attacked the problem *de novo*, and gave a start in obtaining the solutions of the simplest character in the famous theory of three bodies. He showed that there are some comparatively simple periodic orbits in which a small body may move under the influence of two large bodies which are rotating about one another. Hill's methods have been pursued and developed by Poincaré, Darwin, and others. They are of interest on account of the mathematical methods, and also as applied to some small planets and satellites in the solar system.

But the most remarkable progress in Astronomy in recent years has been in our knowledge of the fixed stars. Let us compare them for a moment with the planets. These were to the old astronomers a small number of particularly bright stars, which were remarkable for their peculiar movements. Now, we know a great deal about them. They are bodies more or less like the earth, which revolve round the sun. We know with considerable accuracy their distances and their movements. We know their masses, how they rotate about their axes, and how some of them are accompanied by satellites. We have, in fact, an accurate geometrical knowledge of the present state of the solar system, and clues which enable us to determine its past and future history for a considerable time with certainty, and to speculate on its remote past and future.

We cannot hope to obtain as complete knowledge of the fixed stars as of the solar system, but, nevertheless, a general idea of the geometry of the stellar system is being gradually obtained. Beginning with the stars nearest the sun—among the stars whose distances have been actually measured, about twenty are within the sphere whose radius is one million times the distance of the earth from the sun. Probably

this number represents nearly all the stars in this sphere which are as bright as the 9th magnitude. Thus, to this limit of brightness we know the average distances apart of stars in our immediate vicinity at any rate. A considerable proportion of these twenty stars are double stars, and their masses are consequently determinable. Further, the distances being known, the absolute luminosities as compared with the sun can be obtained. It is found that the masses are all very near that of the sun, but the luminosities vary very greatly. Again, since the angular movements of these stars are known, the resolved parts of their velocities perpendicular to the line in which we see them, are easily determined. For the brighter of these stars, this is supplemented by their velocities to or from us, which is derivable from spectroscopic observations.

For the more distant stars we have generally to be content with more statistical knowledge. By spectroscopic observations the velocities in the line of sight of from one to two thousand stars have been determined. These are not often greater than 100 kilometres a second, and are on the average about 20 kilometres a second. One very remarkable feature is that the blue stars which are in an early stage of development have small velocities: the velocities increase as the stars get older and redder. Further, the actual motion of the sun is derivable from these observations, as stars in one part of the sky are seen to be approaching the sun, and diametrically opposite to be receding from it. When these data are compared with the angular motions of the stars, a determination of average distances is obtained fairly readily.

Another line of research consists in determining the magnitudes of the stars. Here photography has been of great assistance. When the numbers of the stars of different magnitudes are compared, an idea may be obtained of their total number and extent. By the combination of these different classes of observation, it appears that the sun is somewhere near the centre of a finite collection of stars. They extend to more than 1000 million times the distance of the sun, and number perhaps one or two thousand millions. Outside of these there appear to be other groups of stars.

These conclusions have been reached from observations correlated by geometrical and mathematical reasoning. Whether astronomers will have as much success in discovering the order in the stellar system as completely as they have done for the planets is a question which may be asked but can hardly be answered. The problem is a great one, and, I think we shall agree, will provide scope for mathematicians for a long time to come.

F. W. Dyson.

THEORY OF CENTROIDS.

It does not seem to be sufficiently realised that most of the theory usually classed under "Centre of Gravity" is purely geometrical in character; and, on the other hand, such impossible terms as "centre of gravity of an area" still offend the eye. This paper suggests a presentation of the subject free from such defects; but its main object is to establish a sounder transition from the treatment of discrete point-systems, which is the essential groundwork of the theory, to the much further reaching treatment of continuous point-systems. The fact that this transition remains in an unsatisfactory state is an instance of the reluctance to face the necessity for explicit recognition of certain fundamental ideas of the Infinitesimal Calculus, in elementary Mathematics. It is not long since the proper conception of tangent was admitted as suitable for an elementary course of Geometry. Still more

recently the more abstruse conception of arc-length has been faced. From what follows it may appear that the conception of centroid awaits its turn.

1. Define the term

centroid of **n** *points* $P_1, P_2, \ldots P_n$, *with associated* (*real*) *numbers* $p_1, p_2, \ldots p_n$*

as a point C such that its distance ($\bar{D}$) from any plane is related as "weighted mean" to the distances (D_r) of the given points from that plane; so that

$$\bar{D}=(p_1 . D_1+p_2 . D_2+\ldots+p_n . D_n)/(p_1+p_2+\ldots+p_n).$$

This definition involves an "existence-theorem," viz. the theorem that there is a point C which has this general property,—a theorem which may be proved very simply and quite generally, by the method of Orthogonal Projection, as follows:

Specify a point C to have the weighted mean distance from *three* mutually perpendicular planes, intersecting in OX, OY, OZ; so that

$$\bar{x}=\Sigma(p_r . x_r)/\Sigma p_r, \quad \bar{y}=\Sigma(p_r . y_r)/\Sigma p_r, \quad \bar{z}=\Sigma(p_r . z_r)/\Sigma p_r.$$

And let the representative plane of Space be specified to have normal direction inclined at angles α, β, γ to OX, OY, OZ, and to be such that $\mathbf{d}_0$ is the measure of the distance of O from it *in that normal direction.* The distance of P_r from this representative plane is then (using orthogonal projection on a normal to the plane) given by

$$d_r=d_0+\cos\alpha . x_r+\cos\beta . y_r+\cos\gamma . z_r,$$

and $\Sigma(p_r . d_r)=d_0 . \Sigma p_r+\cos\alpha . \Sigma(p_r . x_r)+\cos\beta . \Sigma(p_r . y_r)+\cos\gamma . \Sigma(p_r . z_r)$

$$=(\Sigma p_r) . (d_0+\cos\alpha . \bar{x}+\cos\beta . \bar{y}+\cos\gamma . \bar{z})$$

$$=\bar{d} . \Sigma p_r;$$

$$\therefore \quad \bar{d}=\Sigma(p_r . d_r)/\Sigma p_r.$$

[*N.B.*—The proof, of course, requires no prior knowledge of Analytical Solid Geometry. No harm is done if a few of the ideas of that theory are picked up by the way. For beginners, the analogous process may be first applied to a system of points in a straight line, and then to a system of points in a plane.]

2. (i) From the defining property it immediately follows, by corresponding association and commutation of the two addition-expressions, that *the centroid C may also be specified as the centroid of a system of sub-centroids derived from any system of groups into which the given system of points* (*with their associated numbers*) *may be analysed, each of these sub-centroids having associated with it the sum of the numbers of the group from which it is derived.*

(ii) It is further obvious from the form of the defining expressions that new points may be arbitrarily introduced into the system, without effect upon the centroid C, provided that each point so introduced is used twice and that the two numbers associated with it are opposite† numbers.

3. The particular case of two given points has a special importance. In that case the centroid has the simple property that it is collinear with the given points and divides their join inversely in the ratio of their associated numbers.

Using this fact and a succession of applications of the principle of § 2, (i), we arrive at the unsymmetrical but self-contained determination of the centroid C, by means of points C', C'', C''', ..., such that

C' is the centroid of P_1, P_2 with associated numbers p_1, p_2,
C'' „ „ „ C', P_3 „ „ $(p_1+p_2), p_3$,
C''' „ „ „ C'', P_4 „ „ $(p_1+p_2+p_3), p_4$.
etc.

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* When the associated numbers are all equal, the term "centroid of $P_1, P_2, \ldots P_n$" is used, without further qualification.

† Two numbers may be conveniently called "opposite numbers" if their sum is zero.

It is an immediate corollary that (1) if $P_1, P_2, \dots P_n$ are collinear, C is collinear with them, and (2) if $P_1, P_2, \dots P_n$ are coplanar, C is coplanar with them.

4. (i) The extension to continuous (and therefore infinite) systems of points, constituting a geometrical figure, either (1) linear, or (2) superficial, or (3) "solid," can only be made as follows:

The continuous figure is analysed into **n** elements, according to any law, such that each of the elements is infinitesimal in magnitude (and therefore degenerates to a point) when **n** tends to infinity. A point P_r of the representative element (E_r) is chosen according to any law of dependence upon n, r; and with P_r is associated a number p_{rn} which is the measure of some characteristic quantity obtained from E_r (be it length, area, volume, mass, electric charge, ...).

The system of points (P_r) with their associated numbers (p_{rn}) have a centroid, C_n, dependent in general upon **n** and upon the laws involved in the specification of E_r, P_r.

The extended conception of *centroid* is a consequence of the proposition that *the point C_n tends to a definite limit C when* **n** *tends to infinity, independent of the above-mentioned laws*: a proposition which must—like those which underlie the conceptions of tangent and arc-length—be taken as axiomatic in elementary theory. Its proof, like that of the arc-length proposition, resides in the fundamental theorem of the Integral Calculus.

This limit-point C is called a centroid of the figure; and such terms as "centroid of length, area, volume, mass, etc." may conveniently be used: a centroid of length being obtained from a linear figure by taking p_{rn} proportional to the length of the line-element E_r of that case; a centroid of area from a superficial figure by taking p_{rn} proportional to the area of the surface-element E_r of that case, etc.; and the centroid of mass of a body by taking p_{rn} proportional to the mass of the representative element of the body.

(ii) The indefiniteness of the element E_r and the point P_r can be yet further extended. It is sufficient that E_r should represent elements which (1) are of the same type as the figure (*i.e.* linear or superficial or "solid"), (2) are infinitesimal when **n** tends to infinity, (3) constitute in the aggregate a figure which has the given figure as limit when **n** tends to infinity; and that P_r be a point which tends in the limit to the point to which the element E_r degenerates.

(iii) These facts, of which the exact statement has a highly mathematical flavour, are of the greatest practical importance on account of the freedom they permit in the choice of laws for determination of E_r, P_r so as to obtain the simplest possible investigation of the centroid in a particular case (see § 5 below).

And it is apparent that the specification of centroid coordinates by means of definite integral expressions is of the very essence of the case of continuous systems—being implied in the new definition of centroid which is required for that type of case. It is, in fact, the "existence" of these definite integrals that justifies the extended definition.

5. The facts of §§ 2, 3, above, provide the most useful methods of investigating centroids of geometrical figures. The fundamental cases will be reviewed in this section:

(i) If a geometrical figure be symmetrical about a point, a centroid of length, area, or volume, associated with it, is coincident with the centre of figure. For the figure may be analysed into infinitesimal elements which are symmetrical in pairs about the centre of figure, and are therefore equal in length, area, or volume, as the case may be; and the auxiliary points may be chosen for a pair of elements so as to be equidistant from the centre. Each pair of auxiliary points, with their equal associated numbers, gives a

sub-centroid coincident with the centre. Hence the required centroid is the limiting position of a point C_n which is at the centre of figure for all values of **n**, and therefore has that point as limiting position.

This is the valid argument covering the cases of (1) centroid of length of straight stretch AB, (2) centroids of length and area of complete circle or ellipse, (3) centroids of area and volume of sphere or ellipsoid; also (4) centroids of length and area of parallelogram and (5) centroids of length, area, and volume of parallelepiped.

(ii) Similarly, if a figure be symmetrical about a straight line (or plane), the laws for determination of E_r, P_r can be so chosen that C_n is the centroid of a system of sub-centroids (with their associated numbers) in that line (or plane). Whence the centroid C is itself in that line (or plane).

In the well-known investigation of the centroid of length of a circular arc, this principle is used and also that of § 4, (ii), above,—the elements E_r being most suitably taken as the sides of an equilateral **n**-gon inscribed in the arc, and the points P_r as the mid-points of the corresponding arc-elements.

(iii) Ideas closely analogous to those of (ii) can be neatly applied to the important cases of centroid of area of The Triangle and centroid of volume of The Tetrahedron, although there is not symmetry in these cases.

Analyse the triangle ABC into infinitesimal elements, by radial lines from A dividing BC into equal parts, and parallels to BC at equal intervals from A to BC. The trapezium-elements can obviously be paired so that the elements (E_r, E_r') of a pair are equal in area and can have auxiliary points P_r, P_r' in them chosen so as to be equidistant from the A-median. The auxiliary points of each pair then give a sub-centroid on that median, and the point C_n so determined is therefore on that median for all values of **n** [the possible odd set of elements "down the median" can have their auxiliary points chosen to be on the median]. It follows that the centroid of area of the triangle is on each of its medians; it therefore coincides with the centroid of the three vertices, and divides each median in the ratio 2 to 1.

Exactly the same method may be applied to the tetrahedron $ABCD$ by joining D to the points of the above analysis of ABC and using, further, a set of planes parallel to ABC at equal intervals from D to ABC (*i.e.* analysing the tetrahedron by planes parallel to ABC, planes through D parallel to BC, and planes through AD). The elements may be paired so that the elements E_r, E_r' of a pair are equal in volume, and the corresponding auxiliary points P_r, P_r' chosen equidistant from the plane which passes through AD and bisects BC. It follows that the centroid of volume of the tetrahedron is on the six medial planes, and therefore on the four lines joining the vertices to the centroids of area of the opposite faces; this point is also the centroid of the four vertices and divides each of the lines in the ratio 3 to 1.

(iv) The usual method of investigating the triangle case * is, in generality of principle, the best method, but the usual presentation of it is not sound. The method really depends on the case of the parallelogram, as follows:

By the principle of § 4, (ii), above, the centroid of area of the triangle ABC is the limiting position of the centroid of area of a figure which is constituted of parallelograms with sides parallel to BC and infinitesimal altitudes in such a way that the two figures only differ by a set of triangles of which the aggregate area is infinitesimal. This, of course, leads at once to the result.

To the teacher who may wish to break these ideas gently to his pupils it may be of interest to note that, if the parallelograms be taken to have

* "The C.G. of a uniform triangular plate," is, of course, really the centroid of volume of a prism; and that of "a uniform triangle of rods" is also a centroid of volume. What is usually done under these heads is an investigation of centroid of area and centroid of length of The Triangle—a purely geometrical question.

their other sides parallel to the A-median, the simpler principle of § 4, (i) is sufficient. For the entire triangle is then analysed into (1) a set of parallelograms with centres on the median and (2) a set of infinitesimal triangles which may be paired as E_r elements of the fundamental process so as to give for each pair a sub-centroid on the median.

(v) The centroid of volume of a prism is the centroid of area of its mid-section. This result is most simply obtained by analysing first into a system of prisms of infinitesimal sectional area, and then each of these prisms into a system of elements of equal infinitesimal length. We thus obtain a system of auxiliary sub-centroids located in the several elements into which the mid-section of the prism has been analysed. The numbers associated with these sub-centroids are proportional to the volumes of the longitudinal prisms of the first analysis, and therefore to the areas of the respective elements of the mid-section. Whence the theorem.

(vi) The most elementary investigation of the centroid of volume of The Pyramid uses the investigation, of (iii) above, for the case of the tetrahedron. The pyramid is analysed into a finite number of finite tetrahedra with a common vertex which is the vertex of the pyramid. It follows from the principle of § 2, (i) that the centroid of volume of the pyramid is the centroid of the points which are the centroids of volume of these tetrahedra, with associated numbers proportional to the volumes of the tetrahedra and therefore to the areas of their bases. Whence the required centroid of volume is the centroid of area of the section which is parallel to the base and three-fourths of the way from vertex to base, *i.e.* the point which divides the join of vertex to centroid of area of base in the ratio 3 to 1.

The better method, from the point of view of general principle, is to use the fact that the required point is the limiting position of the centroid of volume of a figure which is constituted of prisms with ends parallel to the base of the pyramid and infinitesimal altitudes in such a way that the two figures only differ by a figure of which the total volume is infinitesimal. The centroids of volume of the prisms are ultimately centroids of area of sections of the pyramid parallel to its base; and it follows that the required centroid of volume is collinear with the vertex of the pyramid and the centroid of area of its base. It may clearly be specified as the centroid of points of that line at equal infinitesimal intervals from vertex to base, with associated numbers in duplicate proportion to distance from vertex—leading to the expression

$$\lim_{n\to\infty} \frac{1}{n} \cdot \{\tfrac{1}{2}\,.\,n\,.\,(n+1)\}^2 / \{\tfrac{1}{6}\,.\,n\,.\,(n+1)\,.\,(2n+1)\}, \text{ which} = \tfrac{3}{4},$$

for the proportional distance of the centroid from the vertex. Or the result may be first applied to the case of the tetrahedron (which is a pyramid in four different ways), and the investigation of the general case then completed as above.

6. This argument may serve to show the true mathematical processes of the theory with which it is concerned. And it may serve the wider object of helping some who may have felt that Integral Calculus methods are rather abruptly and arbitrarily tacked on to elementary theories. It will show that the calculus idea ought to be introduced at an early stage, and that the analytical methods of the calculus will then take their place in the most natural way possible when they are wanted. D. K. Picken.

Wellington, N.Z.

REVIEWS.

The Analysis of Sensations and the Relation of the Physical to the Psychical. By ERNST MACH. Translated from the first German edition by C. M. Williams. Revised and supplemented from the fifth German edition by Sydney Waterlow, M.A. Pp. xvi+380. Cloth, 6s. 6d. net. 1914. (Chicago and London: The Open Court Publishing Co.)

This new, revised, and enlarged edition of Dr. Mach's *Analysis of Sensations* is a book nearly twice as long as the English translation of 1897. The eight chapters of the original edition have all been greatly expanded, and there are six new chapters on "My Relation to Avenarius and other Thinkers," "Physics and Biology," "The Will," "Biologico-Teleological Considerations as to Space," "Sensation," "Memory and Association," and "How my Views have been received." The author amplifies and brings up to date his original discussions on points of detail, and explains and justifies his more general views as to the relation between different branches of science and as to questions on the borderland between science and philosophy. Particular interest attaches to the explanations which he now gives of the way in which his views were developed. This is one of the most important of Mach's works, and the present edition has been most competently translated and revised.

Problems of Science. By FEDERIGO ENRIQUES. Authorised translation by Katharine Royce. With an Introductory Note by Josiah Royce. Pp. xvi+392. Cloth, 10s. net. 1914. (Chicago and London: The Open Court Publishing Co.)

This is a translation of Enriques's *Problemi della Scienza*, of which the first Italian edition appeared in 1906. In the introductory note Professor Royce points out that this work of synthetic scientific methodology "stands somewhat above and apart from those philosophical controversies which the anti-intellectual movement [of James and Bergson] has inspired," and contrasts Enriques's tendency with the tendencies just referred to.

Professor Enriques himself characterises the spirit of this critical study of certain problems relating to the logical and psychological development of scientific knowledge as both "critical and positive." He does not use these words in their usual meaning, but he thinks that he has interpreted things in a clearer and more scientific way, and has reconciled, without eclectic compromises, certain speculative tendencies by which his thought was prompted at the outset (p. xiv).

The first chapter is introductory. There is a reaction at work against that "cowardice of the modern spirit," agnosticism (p. 4). To illustrate this there follows a historical sketch of the "insoluble" problems of the squaring of the circle and perpetual motion. "In a broad sense there are no insoluble problems. . . . There are only problems not yet suitably expressed" (p. 5).

The second chapter, on "Facts and Theories," deals with illusions and reality, and the different kinds of facts that provide the data for the various sciences. The methods and value of scientific theories are then discussed and illustrated by Newton's theory of gravitation, Poisson's electrostatic theory, and the theory of solutions (pp. 86-98). In this chapter the analysis of what constitutes reality is developed into a critique of facts and of theories, so handled as to distinguish between the positive content of science on the one hand, and its subjective aspect on the other.

From this analysis arise two classes of problems which are successively studied: (1) the problems connected with the logical transformation of concepts, regarded both as a psychological development and as an instrument of knowledge (Chapter III.); and (2) those problems which refer to the significance and acquisition of the more general concepts of space, time, force, motion, and so on (Chapters IV. and V.). In Chapter III. are discussed, among other things, symbolic and psychological logic, the relation of logic to mathematics, the definitions of various kinds, and the processes and principles of logic. The second part of this chapter is devoted to the application of logic and a discussion of the value and limits of logical principles. The fourth chapter is on geometry, and contains much critical, historical, and psychological information from one of the most competent men of the present day. The fifth chapter treats mechanics in much the same way.

The theoretical questions of physics are examined in Chapter VI. in connection with the "Extension of Mechanics." Here are very useful parts on the principle of relativity and the extension of mechanical explanations into the field of the phenomena of life.

There is a misspelling of Van der Waals's name on p. 35; on p. 106, the date after Frege's name should be either 1879 or 1893, not 1891; the statement on p. 293 that Lagrange observed the connection of the principle of least action with Gauss's principle is mistaken. There seems to be a disregard on p. 105 of the fact that particular propositions are existential, whereas universal ones are not. There seems to be a certain superficiality, like Bergson's, in the thesis (p. 146) "that the properties of assemblages of objects require a certain invariability in the latter." Arithmetic does not cease to be applicable when globules of mercury coalesce into one. The "limits of the application of logic" would seem, according to Enriques, to be determined by psychological circumstances, such as overeating or headaches.

But these are only small defects. The book is able, interesting, and important. The importance is as great for teachers as it is for philosophers.

Das Relativitätsprinzip. Von H. A. Lorentz. Drei Vorlesungen gehalten in Teylers Stiftung zu Haarlem. Bearbeitet von W H. Keesom. Beihefte zur Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, herausgegeben von W. Lietzmann und E. Grimsehl. Nr. 1. Pp. 52. Stout paper covers, 1.40 marks. (Leipzig and Berlin: B. G. Teubner.)

Einstein's principle of relativity is that phenomena in a system of bodies depend only on the positions and motions of those bodies with respect to one another; in the sense that a constant motion of translation affecting the system as a whole has no influence on the phenomena taking place in the system. The connection of this principle with the questions about an aether is very clearly put, and an excellent account of the work of Fitzgerald, Lorentz, Einstein, Minkowski, and many others is given.

Das Relativitätsprinzip: eine Einführing in die Theorie. Von A. Brill. Zweite Auflage. Pp. ii + 34. Stout paper covers, 1.20 marks. (Leipzig and Berlin: B. G. Teubner.)

Some improvements and additions have been made to the second edition of this good and simple discussion of the principle of relativity in pure mechanics.

Our Knowledge of the External World as a Field for Scientific Method in Philosophy. By Bertrand Russell. Pp. x + 245. Cloth, 7s. 6d. net. 1914. (Chicago and London: The Open Court Publishing Co.)

These eight "Lowell Lectures," delivered at Boston (Mass.) in March and April 1914, attempt to show, by means of examples, the nature, capacity of limitations of the logico-analytical method in philosophy, which, in the author's opinion, yields whatever scientific knowledge it is possible to obtain in philosophy. "The central problem," says the author, "by which I have sought to illustrate method is the problem of the relation between the crude data of sense and the space, time, and matter of mathematical physics." Many of the difficulties between the views advocated here and those of *The Problems of Philosophy* have been raised by Dr. A. N. Whitehead; and much of these lectures is a rough and preliminary statement of what Dr. Whitehead will say in the fourth volume of Dr. Whitehead's and Mr. Russell's *Principia Mathematica*. The author's chief debts are to G. Frege, on logic, and G. Cantor, on the mathematical infinite.

Lecture I., on "Current Tendencies," distinguishes three types in present-day philosophies: (1) The classical tradition; (2) Evolutionism, now represented by Nietzsche, William James and Henri Bergson; (3) The author's "logical atomism," which is allied to "the New Realism," and has a logical and mathematical origin. The first two types are criticised; the former partly in connection with Mr. F. H. Bradley's *Appearance and Reality*; the latter chiefly in connection with M. Bergson's philosophy.

Lecture II. is on "Logic as the Essence of Philosophy"; and shortly describes Aristotelian and mediaeval logic, the method of induction, Hegel's "logic," and the mathematical logic of Leibniz, Boole, Frege, and Peano. It explains the notion of the "form" of a proposition, and shows that this form alone is essential in all

inference. It also shows, by the help of the classification used in the logic of relations, that not all relations can be reduced to the predications of traditional logic, and concludes with a discussion of logic, empirical facts, and the intermediate region.

Lecture III. is "On our Knowledge of the External World," and applies the logical-analytic method to the problem referred to. Sense-data are the sole basis of our knowledge of the external world.

Lecture IV. considers the discrepancy between "The World of Physics and the World of Sense," and finds it to be more apparent than real. Mr. Russell shows that whatever there is reason to believe in physics can probably be interpreted in terms of sense. At this point is given an illustrative example, due to Dr. Whitehead, of how a definition of a mathematical point can be given in terms of sense-data. Mr. Russell defines series or classes of sense-data which can be called particles, points and instants. If such constructions are possible, then mathematical physics is applicable to the real world in spite of the fact that its particles, points, and instants are not to be found among actually existing entities; and here the works of Poincaré (*Science et hypothèse*) and Mach (*Analysis of Sensations*) are mentioned. Finally, Russell's "Principle of Abstraction" is discussed.

Lecture V. is on the "The Problem of Infinity considered Historically." The explanation of the physical world which assumes infinity and continuity is easier and more natural than any other, and, since the supposed difficulties of continuity all reduce to difficulties concerning infinity, in freeing the infinite from contradiction (as, in essentials, Georg Cantor has done) we are at the same time showing the logical possibility of continuity as assumed in science. After discussing very thoroughly and rejecting Kant's first two antinomies, which are examples of the way in which infinity has been used to discredit the world of sense, Mr. Russell states and explains the problem of infinity, shows how it arose with the Pythagorean discovery of incommensurables, and shows the irrelevance of all the solutions proposed by philosophers from Zeno to Bergson. Here a valuable contribution is made to the elucidation of the arguments of Zeno and their relation to the doctrines of the Pythagoreans. Still more strongly than Paul Tannery, Mr. Russell says that Zeno's arguments "are not . . ., on any view, mere foolish quibbles: they are serious arguments, raising difficulties which it has taken two thousand years to answer, and which even now are fatal to the teachings of most philosophers. . . . Zeno's arguments, in some form, have afforded grounds for almost all the theories of space and time and infinity which have been constructed from his day to our own. We have seen that all his arguments are valid (with certain reasonable hypotheses) on the assumption that finite spaces and times consist of a finite number of points and instants, and that the third and fourth almost certainly, in fact, proceeded on this assumption, while the first and second, which were perhaps intended to refute the opposite assumption, were in that case fallacious. We may therefore escape from his paradoxes either by maintaining that, though space and time do consist of points and instants, the number of them in any finite interval is infinite; or by denying that space and time consist of points and instants at all; or lastly, by denying the reality of space and time altogether. It would seem that Zeno himself, as a supporter of Parmenides, drew the last of these three possible deductions, at any rate in regard to time. . . . But . . . the difficulties can also be met if infinite numbers are admissible." Indeed, if we are to solve all the difficulties derivable from Zeno's by analogy, we must discover some tenable theory of infinite numbers, and the difficulties which philosophers have found in the notion of infinity are then discussed; such as that "there cannot be anything beyond the whole of an unending series."

Lecture VII., on "The Positive Theory of Infinity," gives a lucid account of Georg Cantor's theory of infinity and Gottlob Frege's and Mr. Russell's own work on the definition of number, etc. New and very important considerations (partly due to Mr. Ludwig Wittgenstein) on the formal nature of the "logical constants" are indicated.

In Lecture VIII., "On the Notion of Cause, with Applications to the Free-Will Problem," there is first a discussion of causality, in which Mr. Russell inquires: (1) What is meant by a causal law; (2) What is the evidence that causal laws have held hitherto; (3) What is the evidence that they will continue to hold in the future; (4) How the causality which is used in science differs from that of common-sense and traditional philosophy; (5) What new light is thrown on the question of

free will by this analysis of the notion of cause. Finally, Mr. Russell tries to reach a general account of the logical-analytical method of scientific philosophy, and a tentative estimate of the hopes of philosophical progress which it allows us to entertain.

These lectures are written with that clearness, force, and subtle humour that readers of Mr. Russell's other works have learnt to expect. But it is also because these lectures contain the first published indication of the new investigations to which the logical and analytical school, of which Mr. Russell is the most prominent exponent, is now tending, that they must be read and studied by all who are interested in science or philosophy, or scientific philosophy.

PHILIP E. B. JOURDAIN.

Handbuch der angewandten Mathematik herausgeben von H. E. TIMERDING. Zweiter Teil: **Darstellende Geometrie** von Dr. JOHANNES HJELMSLEV. Pp. ix + 320. 5 m. 40 f. 1914. (Teubner, Leipzic.)

The subject of Descriptive Geometry has grown from the needs of problems occurring in Art and Technical work; hence the inclusion of this volume in a Handbook of Applied Mathematics. The editor claims that the book serves the double purpose of giving the practical man opportunity to deepen his theoretical knowledge and of leading the mathematician to a greater interest in practical problems. But, the book's appeal is decidedly to the Pure Mathematician; the treatment is entirely theoretical, and the few practical problems that are presented are introduced solely for the special geometrical principles they involve, and not at all on account of their utility. Care has been taken to exclude all technical details, the introduction of which would have tended to obscure the main purpose of the book.

The subjects dealt with and their order of presentation are of interest to the English reader. After a preliminary chapter giving some elementary theorems of solid geometry and simple projection, there is a chapter dealing with double projection, following Monge, then comes isometric projection followed by central-projection and the method of perspective representation, with sections on Cross Ratios and figures in perspective. The next chapter deals with projective ranges and pencils; in this we have Pascal's and Brianchon's theorems for the circle involution, pole and polar properties, and a short section on the principle of duality. The conic sections are then obtained as sections of cones in the ordinary way; Pascal's and Brianchon's theorems are generalised, and the method of drawing a conic through five points indicated. Special properties of the ellipse, parabola and hyperbola follow, and the chapter closes with a section on circles of curvature. The next chapter deals with some general properties of plane curves—involutes, evolutes, roulettes, cycloidal curves and envelopes. The remainder of the book is solid geometry: solids of revolution, surfaces of the second degree, curves in space, developable and enveloping surfaces, with short notes on such special surfaces as the conoid, cylindroid, screw surfaces and contours. There is a final chapter on the construction of shadows and the illumination of surfaces, but this is very brief and of no great practical value.

This list of contents will suggest that the book is somewhat of a "hotch-potch," and indeed, in parts, it is necessarily scrappy. But its extensive scope is made possible mainly by the adoption of a style which, though sufficiently concise to make serious demands on the thought of the reader, is yet sufficiently clear to be readily intelligible.

The second half of the book shows clearly how much more powerful, natural and complete is the geometrical method of treating the solid geometry than the analytical method depending on the use of infinitesimals.

This volume forms an admirable introduction to modern geometrical methods; it is compact and clear; the diagrams are excellent and, as they are not overloaded with detail, are a real aid to the text.

Its value as an introduction is increased by the addition of bibliographies at suitable intervals. It is, perhaps, sufficient indication of the place this subject has occupied in English mathematical education in the past to say that only two English books appear in these lists, viz.:

A. Sopwith, *Treatise on Isometrical Drawing*. London, 1834.

Br. Taylor, *New Principles of Linear Perspective*. London, 1719. A. DAKIN.

Analytische Geometrie der kubischen Kegelschnitte. Von Dr. Otto Staude. Pp. viii + 242. 9 m. 1913. (Teubner.)

The publication, within a few months of one another, of *The Twisted Cubic*, by P. W. Wood, M.A. (Cambridge Tracts in Mathematics, No. 14), and of the volume under review provides the student of analytical geometry with easy access to many interesting and important theorems connected with cubic curves in space. The mathematical equipment demanded of a reader of these volumes is not so great as to prevent their extensive use, with the result that greater attention will doubtless be paid and further research directed to the space-cubic. Each book consists of two parts, one dealing with projective, and the other with metrical properties of the curves. Prof. Staude, unlike Mr. Wood, takes the metrical properties first. He chooses rectangular axes and considers the curve of intersection of a cone and cylinder of the second degree which have a common generator, showing that the curve is a cubic curve. The various cubics that arise when the cylinder is elliptic, hyperbolic, or parabolic are then examined and classified according to the coincidence or otherwise of their points at infinity. Four types of cubic are shown to exist: the cubic ellipse, the cubical hyperbola, the cubical hyperbolic parabola, and the cubical parabola. It is then proved that the general cubic curve, represented in the usual way by parametric equations, belongs to one of the above four types, and various associated loci, such as osculating planes and circumscribing quadrics of revolution, are discussed. The remainder of the first part of the book is occupied with an account of the special properties of each of the four types of cubic. It contains very careful and clear diagrams of the curves, which are of great assistance to the reader.

In the second part of the book we have a very full treatment of the projective properties. Tetrahedral coordinates are taken and the properties of various surfaces, complexes, and tetrahedra connected with the curve are discussed. The last chapter deals with projective methods of generating the cubic.

Many references to original memoirs are given, and an index is appended.

The author is to be congratulated on having produced such an attractive and valuable treatise. R. J. T. B.

A First School Calculus. By R. Wyke Bayliss. Pp. 180 + 103 pp. Answers. Cloth 4s. 6d. (London, Arnold.)

This book is an attempt to introduce the Calculus by means of simple questions based upon concrete problems. The instrument necessary to solve the problem is constructed by means of a series of questions and suggestions of the following type.

P. 7. The formula for the distance s feet fallen by a stone in the time t secs. is $s = 16t^2$ approximately. It is required to find the velocity at any time.

(i) If in the additional time Δt it has fallen through an additional distance Δs, write down the equation connecting $s + \Delta s$ and $t + \Delta t$.

(ii) By subtracting find the relation between Δs and Δt.

(iii) Hence find the average value of $\dfrac{\Delta s}{\Delta t}$.

(iv) What is the average velocity of the stone during the interval Δt?

(v) Find the average velocity during the $10''$ after t''.

(vi) Find the average velocities during $1''$, $\frac{1}{10}$ sec. and $\frac{1}{100}$ sec. *after* t''.

(vii) Find the average velocity during $\frac{1}{1000}$ sec. *after* falling for $10''$.

(viii) How could we discover the actual velocity *at* the end of the tenth second from the start?

(ix) How can we discover the *actual* velocity at the *end* of the time t, *i.e.* at the beginning of the interval Δt?

The book consists of 139 pages of such investigations, covering an elementary Course of Differential and Integral Calculus, then a short introduction to Taylor's Theorem on similar lines, followed by a good selection of miscellaneous examples. The rest of the book, about 100 pages, consists of the Answers to the multitude of questions of the type illustrated above.

The method adopted by Mr. Bayliss is that which many teachers would use when teaching the Calculus to a class of beginners, but it is doubtful whether

it is to be recommended as the proper treatment in a text book. The Socratic method instructs by exposing the ignorance of the pupil, but if the teacher's place is taken by the text book, the method seems more likely to conceal than to reveal this ignorance. The first investigation in the book discusses the speed of a cyclist travelling with variable velocity, and the pupil is asked to show that in order to determine the *exact* speed of the cyclist at a given instant, we must measure the distance he goes in an indefinitely small interval of time, although no explanation is given of the meaning of the phrase "*exact* speed," in fact, there is no discussion as to the meaning of a limit until the Integral Calculus is attacked.

It is unfortunate that the index should be placed before the Answers in the middle of the book, and it would greatly facilitate reference to the Answers if the pages on which the questions occur were inserted. R. C. F.

First Year Course in Mathematics. (Geometry and Trigonometry.) By Prof. K. J. Sanjana. Pp. 240 + xv (tables). 1.12 r. (K. & J. Cooper, Bombay.)

This book has been written specially for the use of students reading for the First-Year Course in Arts of the Bombay University. It presupposes a knowledge of the Matriculation Course in Geometry, *i.e.* approximately Euclid I.-III. 34.

The first four chapters complete what is usually considered the modern equivalent of Euclid I.-IV. Chapter V. deals with Ratio and Proportion; Chapters VI., VII., complete the equivalent to Euclid VI.; Chapter VIII. contains sections on Harmonic Section, Pole and Polar, etc. Then follow eighty pages giving a very good exposition of the "Trigonometry of one Angle," including an explanation of Logarithmic and other tables. The book concludes with a set of tables.

The book should be exceedingly useful for the purpose designed; and, although both the setting and the arrangement will be unusual to teachers in English schools, it will well repay a careful reading. There is certain flavour of originality pervading the work; many of the proofs are distinct advances on those more generally accepted; and its collection of Exercises, theoretical and practical (many of which are worked out as models for the private student) would be hard to match.

Outdoor and Indoor Arithmetics. (Teacher's Book.) By H. H. Goodacre, E. F. Holmes, C. F. Noble, and Percy Steer. Pp. 373. Price 3s. 6d. net. (Bell.)

This book contains the five parts of the Experimental Arithmetic published by the authors and already noticed in these columns, with notes for the teacher's use printed on alternate pages. These notes are uniformly interesting and should be very useful. There is one point, however, which might well have been given further consideration. On page 189 the teacher is directed to explain the difference between Geographical and Magnetic North: the compass described on page 363 points to Magnetic North, but nothing is said of the usual Mariner's compass card which swings with the needle, or of an arrangement of a pointer to give Geographical North or East, whilst a concealed needle attached to it points Magnetic North.

1. **Arithmetic.** By W. M. Baker and A. A. Bourne. In two parts, 2s. each, or complete 3s. 6d. With or without answers. Pp. 400. (Bell.)

2. **Arithmetic.** By H. Freeman. Pp. 224 + tables and answers. Price 2s. 6d. (Bell.)

3. **Arithmetic.** By N. J. Chignell and W. E. Paterson. Part I. Pp. 317 + answers (or without). Price 2s. 6d. Part II. Pp. 265 + tables + answers (or without). Price 2s. 6d. Complete, with or without answers. Price 4s. 6d.

Examples from above, 2s. each part, or 3s. 6d. complete. (Clarendon Press.)

1. Messrs. Baker and Bourne's book should prove very useful. The strongest feature is the careful grading of the exercises. One of the weak points in most arithmetics, fractions, is here made good by the frankly explanatory nature of the exposition of the rules; we have, however, the usual unsatisfactory explanation of negative numbers by means of cows and pounds. "If a man gains £6 and loses £10, he has £4 less than at first" is not symbolically represented by

"£6 − £10 = −£4."

If he had nothing to start with, there is no meaning to be attached to the statement; the man is bankrupt and cannot pay. If he had £P to start with (>£4), we have

$$£P+£6-£10=£P-£4=£P+£(-4) \text{ perhaps.}$$

Any way it explains, if anything, the *addition* of $-£4$, but certainly not the *meaning* of $-£4$. "Any fool of a boy" can understand the necessity of the invention of negative numbers as numbers below zero and the necessity for them to obey the rules already found out for ordinary numbers, and with a little coaxing and jogging can follow the generalising to literal symbols, and *afterwards* the application to practical problems on the thermometer, credit and debit, etc. But the reverse method never leaves the pupil sound on his fundamentals. Besides any student, let alone a boy, loses his grip of the subject when he is continually breaking away for "explanations," which are far harder to grasp, as logical reasons, than the clear statements one makes about pure numbers.

The authors are very wise in giving "limits of error" as their fundamental instead of "percentage or relative error." Those who have external examining to do will appreciate the practical impossibility of teaching a young student the correct use of relative error.

The section on logarithms is good, but the authors slur the fact that a logarithm is not "a fraction," in general, but an incommensurable. The addition of a paragraph (such as is given by Prof. Bowley) explaining the approximate nature of a logarithmic index would have sufficed.

2. Mr. Freeman's book is not of such an exhaustive character as either No. 1 or No. 3 in this list, and perhaps does not suffer in any degree from this fact. It consists mainly of examples, with good hints and notes on method. There is a splendid lack of artificial examples. There is a serious attempt to improve the usual character of chapters on approximate methods. For all that it is hardly satisfactory.

To start with, he rightly urges the use of the rough check: that being so, what use is there in writing the multiplication of 792 by 368, with the 3 under the 2 as

$$\begin{array}{r} 792 \\ 368 \end{array} \left\{ \begin{array}{c} \text{instead} \\ \text{of} \end{array} \right\} \begin{array}{l} 792 \\ 368\,? \end{array}$$

One is glad to see that he insists on multiplication by the "highest" figure first. This should be taught in every school (*versus* Prof. Smith). But the explanation is not sufficiently full, especially in the case of Relative Error, and the more important absolute error which has to be satisfied with one line. The examples, however, on the section are very good. The chapter on Logarithms is too scrappy to be of any use, and should be replaced by a chapter on the "Use of Logarithmic tables," leaving all explanations of logarithms as indices of powers of 10 to later work in Algebra. It would have been better to have grouped the chapters on Interest, etc., with the chapter on Proportion, of which they are all particular cases. It is not sufficiently recognised that Arithmetic consists of Elementary Rules, Proportion and Common-sense, and the greatest of these three is Common-sense.

The Test Papers and Miscellaneous Examples are excellent.

3. Messrs. Chignell and Paterson set out with the laudable object of placing the subject-matter of Arithmetic before the average student in a logical sequence, whilst reducing to a minimum the memorising of rules, and go very far on their way to this desired goal.

Even the first 70 pages give food for reflection, the left to right method of multiplication, the idea of local value, with the almost immediate introduction of decimals, the generalisation to literal numbers (p. 40)—some old-fashioned teachers will stare,—lastly, the introduction by a perfectly natural method of 10^{-1} as a *symbol* (p. 54) and "CORRECT TO FIVE SIGNIFICANT FIGURES" (p. 63), all in such a natural, logical, and *therefore* easy, way that I should have liked to have seen these pages winding up with expression of a number in standard form.

There is, however, the usual unsatisfactory consideration of fractions by dividing the unit; this could have been obviated by using the dodge of a sub-unit: but why not go on as they have begun, logically? I do not like this chapter at all, although the authors point out, what so few seem to know, the difference between

"of" and × after a ÷ sign. Ex. (1) worked out on p. 170 is wrong, the method is not good, and such sums are futile.

Under the heading "Application of Ratio Methods," the authors give a very sound exposition of what I have always known as the "abbreviated unitary method."

Although percentage error is given, it is in its proper place, *after* absolute error: hardly enough work is given on this important section.

There is a very fine set of Miscellaneous Problems at the end of the second part, and the Revision Papers are good and numerous.

Mathematical Problem Papers for Secondary Schools. By Dr. Davison. Pp. 175. Price 2s. (Bell.)

An excellent collection of 200 Papers, each consisting of eight exercises, with an appendix of fifty standard riders in Geometry. Many of the exercises come from public examination papers, but there are a goodly number of the better class that are original. There is a slight leaning, however, to the problem type rather than the exercise type. On the whole, should be exceedingly useful.

Elementary Geometry, Theoretical and Practical. Vol. I. By W. E. Paterson and E. O. Taylor. Pp. 155. Price 1s. 6d. (Clarendon Press.)

This volume deals with the properties of triangles and quadrilaterals, with a brief reference to Proportion and similar triangles. The method is a good one. Each proposition is considered, more or less, as a rider on previous propositions, and is led up to in a chatty manner before its enunciation is stated; in other words, in exactly the manner which an experienced teacher would proceed when using, say, Todhunter's *Euclid*. A very good point is the frequent recapitulation of what has gone before, enabling the student to fix in his memory "what tools he has in his box," for rider work. This method of producing problem power is still further elaborated by giving hints only, but sufficient hints, for the proofs of later propositions.

I hardly like the concealed algebraic nature of the proofs of Euc. II. 12, 13, although, and because, they are founded on Euc. II. 4, enunciated as "The square on the sum of two lines, etc." If an algebraic proof is given, it should be frankly algebraic. I do not think many examiners would accept

$$(AB+BC)^2=AB^2+2AB\,.\,BC+BC^2.$$

The same may be said, and with greater reason, about the authors' definition of Parallel Straight Lines, as those drawn in the same direction. The idea of parallelism is fundamental before the concept of direction can be arrived at by the boy. Also no mention is made of "the same plane." J. M. Child.

Essays on the Life and Work of Newton. By Augustus De Morgan. Edited, with Notes and Appendices, by Philip E. B. Jourdain. Pp. xiv+198. Cloth, 5s. net. 1914. (Chicago and London: The Open Court Publishing Company.)

Augustus De Morgan's biographical sketch entitled "Newton" appeared in *The Cabinet Portrait Gallery of British Worthies* in 1846, and is the first essay printed in this volume. It was, after Baily's *Life of Flamsteed* of 1835, the first English work in which the weak side of Newton's character was made known. Justice to Leibniz, to Flamsteed, even to Whiston, called for this exposure; and the belief that it was necessary did not lower the biographer's estimate of Newton's scientific greatness, and of the simplicity and purity of his moral character. Francis Baily's discovery of the correspondence between the Rev. John Flamsteed, the first Astronomer Royal, and Abraham Sharp, as well as between Newton, Halley, and Flamsteed, on the publication of Flamsteed's catalogue of stars, had thrown a new light on the character of Newton. It appeared that the practical astronomer had been treated ungenerously by Newton, who failed to observe the conditions of publication agreed to by all parties; and afterwards, when remonstrated with, omitted the name of Flamsteed in places where it had formerly stood in the earlier editions of the *Principia*.

It was not only mathematical discovery and controversy that De Morgan treated in the just, broad-minded, and high-minded way that is characteristic of him. He disclaimed any particular interest in those religious beliefs of Newton which he discussed so thoroughly; still, he seems to have felt more interest in the

question, from its own nature, than he was himself aware of. He said, "Whatever Newton's opinions were, they were the result of a love of truth, and of a cautious and deliberate search after it." That Newton was a firm believer in Christianity as a revelation from God is very certain, but whether he held the opinions of the majority of Christians on the points which distinguish Trinitarians from Arians, Socinians, and Humanitarians, is the question of controversy.

The second of De Morgan's Essays printed in this volume concerns the great controversy about the invention of the fluxional or infinitesimal calculus, in which Newton and Leibniz were the principals. The Essay printed is from the *Companion to the Almanac* of 1852, and is now extremely rare. It is of great interest and importance, both on account of the fairness and vigour which De Morgan always showed in the defence of Leibniz against the imputations of Newton and the Royal Society, and because it first introduced the English public to Gerhardt's important discovery of Leibniz's manuscripts, showing his gradual discovery of the calculus in 1673-1677. This Essay also contains a summary of much of De Morgan's historical work on the controversy. Where it seems advisable, notes have been added to the second Essay, giving an account of De Morgan's and others' work on the subject.

To this second Essay the editor has added an appendix, the chief aim of which is to give the sources at which may be found the original manuscripts written by Newton and Leibniz when they were discovering their respective calculuses. This has not been done hitherto, and it is all the more necessary that it should be done, as modern authors, such as Moritz Cantor in his *Vorlesungen über Geschichte der Mathematik*, neglect the fact that any early manuscripts of Newton's on fluxions are extant, or that some have been published—by Rigaud, for example—and that some still remain unpublished.

In 1855 appeared Sir David Brewster's *Memoirs of the Life, Writings, and Discoveries of Sir Isaac Newton*, and De Morgan, in a critique of this work in the *North British Review*, showed clearly that Sir David had fallen into hero-worship. Here the faults of Newton are pointed out with an unwavering finger, and the merits of Leibniz are recognised and his character defended against Brewster more at length than in De Morgan's biography of Newton. This review is printed as the third of De Morgan's Essays on Newton. The editor has added two appendices to this third Essay: the first is part of a biography of Leibniz which De Morgan wrote, and which illustrates a laudatory reference to that great man in the third Essay; the second is an extract from a later work of De Morgan's, and deals with Newton's character and the relation to it of the Royal Society down to De Morgan's own times.

Numerous notes of either a bibliographical, explanatory, or critical nature have been added to all the Essays, but all that is not De Morgan's is put in square brackets. Such notes have become necessary, and it is hoped that the present ones will reply to all the calls of necessity and will make the book both useful and complete. Very little has to be criticised in De Morgan's history or conclusions. Like everything he wrote, these Essays of his are marked by scrupulous care, sanity of judgment, and wide reading; and one hardly knows which to admire most: the breadth or the height of his mind.

The frontispiece is from an engraving by E. Scriven of Vanderbank's portrait of Newton in the possession of the Royal Society of London. An engraving from this picture accompanied the original of De Morgan's biographical sketch; but the present frontispiece is from a much finer engraving prefixed to the biography of Newton in the first volume of *The Gallery of Portraits: with Memoirs* of 1833.

X.Y.Z.

QUERIES.

(92) When Lagrange uses *quantité de mouvement*, does he mean momentum or energy?

(93) Who first wrote M for $\frac{W}{g}$? Was it Euler?

THE LIBRARY.

The Librarian acknowledges, with thanks, the receipt of *Arithmetic*, by Chignell and Paterson, and *Introduction to the Infinitesimal Calculus*, by Caunt, both presented by the Clarendon Press. Also of six volumes issued by the Open Court Publishing Co., and presented by Dr. Carus.

The Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue has been issued to members containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Wanted by purchase or exchange:

1 or 2 copies of *Gazette* No. 2 (very important).
1 or 2 „ „ No. 8.
2 or 3 copies of Annual Report No. 11 (very important).
1 or 2 „ „ Nos. 10, 12 (very important).
1 copy „ Nos. 1, 2.

ERRATUM.

Vol. vii. No. 110, March, 1914, p. 274, l. 2, *for* − *read* +.

BOOKS, ETC., RECEIVED.

The Teaching of Mathematics in Australia. (Report presented to the International Commission on the Teaching of Mathematics.) By H. S. Carslaw. Pp. 79. 1914. (Angus & Robertson, Sydney.)

Über die mathematische Erkenntnis. By A. Voss. 3rd edition. Pp. vi + 148. 5 m. 1914. (Teubner.)

Les Coordonnées intrinsèques. Théorie et Applications. By L. Braude. Pp. 100. With portrait of E. Cesàro. (Scientia, No. 34.) 2 frcs. 1914. (Gauthier-Villars.)

A Geometrical Vector Algebra. By T. Proctor Hall. Pp. 30. 1914. (Western Speciality, Ltd., Vancouver, B.C.)

Elementary Geometry. Theoretical and Practical. Vol. I. *Triangles and Quadrilaterals.* By W. E. Paterson and E. O. Taylor. Pp. 160. 1s. 8d. 1914. (Clarendon Press.)

John Napier and the Invention of Logarithms, 1614. By E. W. Hobson. Pp. 48. 1s. 6d. net. 1914. (Camb. Univ. Press.)

Problems of Science. By F. Enriques. Pp. xvi + 392. 10s. net. 1914. (Open Court Co.)

New Concrete Practical Arithmetic Tests. Edited by J. L. Martin. In 5 books. 72 pp. each. Sewed, 4d. each; Limp cloth, 5d. each. Ans. 3d. each net, or in 5 sets of 36 cards each in case, with Answers, 1s. 3d. net each. 1914. (Harrop & Co.)

A Course of Geometry. Theoretical and Practical. A Class Book for Secondary and Technical Schools. By A. H. Bell. Pp. vii + 127. 2s. 6d. 1914. (Rivington.)

Percentage Trigonometry, or *Plane Trigonometry reduced to Simple Arithmetic.* By J. C. Ferguson. Pp. 155. 3s. 6d. net. 1914. (Longmans, Green.)

Pendlebury's New Concrete Arithmetic. By C. Pendlebury and H. Leather. Yrs. 1-3, 4d. each; yrs. 4, 5, 6d. each; or in paper, 1d. less per yr. 1914. (Bell & Sons.)

Matriculation Mechanics. By W. Briggs and G. H. Bryan. 9th impression. (3rd edition.) Pp. viii + 363. 3s. 6d. 1914. (Univ. Tutorial Press.)

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